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# SEPARATING POINTS FROM CLOSED CONVEX SETS OVER ORDERED FIELDS AND A METRIC FOR $\tilde{R}^n$

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ABSTRACT. Let $R$ be an arbitrary ordered field, let $\overline{R}$ be a real closure, and let $\tilde{R}$ and $\tilde{R}^n$ denote the real spectra of $\overline{R}[X]$ and $\overline{R}[X_1, \dots, X_n]$. We prove that a closed convex subset in $R^n$ may be separated from a point not in it via a continuous "linear" functional taking values in $\tilde{R}$ and that there is a $\tilde{R}$-valued metric on $\tilde{R}^n$. The methods rely on the ultrafilter interpretation of points in $\tilde{R}^n$ and on the existence of suprema and infima of sets in $\tilde{R}$.

## INTRODUCTION

A basic result which uses the completeness and order relation on $\mathbb{R}$ is

**Theorem I** (for $\mathbb{R}$). *Let $C \subset \mathbb{R}^n$ be a closed convex set and let $\mathbf{p} \in R^n \setminus C$. Then there exists a linear function*

$$L(\mathbf{x}) = \mathbf{a} \cdot (\mathbf{x} - \mathbf{b})$$

*such that $L(\mathbf{p}) < 0$ and $L(\mathbf{q}) > 0$ for all $\mathbf{q} \in C$.* □

Theorem I remains true if $\mathbb{R}^n$ is replaced by a locally convex real vector space and $\mathbf{p}$ is replaced by an arbitrary compact set [5, 3.3, Satz 4] but we are more interested in the remark that the *statement* of Theorem I makes sense if $\mathbb{R}$ is replaced by any ordered field $R$. It is false as stated for *every* $R \neq \mathbb{R}$, but if we replace $L$ by an $\tilde{R}$-valued linear function where $\tilde{R} := \operatorname{Sper} R[X]$ is the *real spectrum* of $R[X]$, we obtain

**Theorem I.** *Let $R$ be any ordered field, let $C \subset R^n$ be a closed convex set, and let $\mathbf{p} \in \mathbb{R}^n \setminus C$. Then there exists a continuous $\tilde{R}$-valued linear function $L : R^n \to \tilde{R}$ such that $L(\mathbf{p}) < 0$ and $L(\mathbf{q}) > 0$ for all $\mathbf{q} \in C$.*

In order to prove Theorem I for general $R$, we let $\overline{R}$ be a real closure of $R$ and extend the Euclidean norm to the space

$$\tilde{R}^n := \operatorname{Sper} \overline{R}[X_1, \dots, X_n],$$

which may be thought of as consisting of those points in $\operatorname{Sper} R[X_1, \dots, X_n]$ which induce the given order on $R$. This extended norm does not give us

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a metric on $\widetilde{R}^n$ because $\widetilde{R}^n$ is not equipped with an addition, and we cannot define an inner product on $\widetilde{R}^n$ as is discussed after Definition 2 in §3. However, with virtually no fudging of the definitions, we are able to prove

**Theorem II.** *Let $R$ be any ordered field. Then there is a metric*

$$\mu : \widetilde{R}^n \times \widetilde{R}^n \to \widetilde{R}$$

*which extends the Euclidean distance function.*

The point of this paper, then, is that the real spectrum provides the correct tool for both compactifying and completing general ordered fields with a view towards preserving at least some of the structure familiar from linear analysis over $\mathbb{R}$. We assume a certain degree of familiarity with the spaces $\widetilde{R}$ and $\widetilde{R}^n$. In particular we assume that the reader is familiar with

(1) the various types of points in $\widetilde{R}$ for general real closed $R$,
(2) the description given in [7 §2]
(3) the "ultrafilter theorem" [3] and ultrafilter arguments, and
(4) semialgebraic maps $f : R^n \to R$ and their extensions $\widetilde{f} : \widetilde{R}^n \to \widetilde{R}$.

All of the necessary background is available in [2, Chapter 7], [1], and [7, §2].

The paper is organized as follows: In §1 we present a simple counterexample showing why Theorem I is false if we only allow $R$-valued linear functions, and in §2 we present an elementary proof of Theorem I for $\mathbb{R}$. In §3 we explain what we mean by a $\widetilde{R}$-valued linear function and prove a continuity result for these functions. In §4 we isolate two results needed to modify the proof given in §2, and in §5 we prove Theorem I for general $R$. In §6 we generalize the notion of "slices" from [7] to one called "rips". Rips allow us to add points in $\widetilde{R}^n$ and to compare the resulting sums so that we may interpret the triangle inequality. In §7 we define $\mu$ and prove Theorem II.

*We point out right away that, for the purposes of our proofs, we will assume that $R$ is real closed.* This is no loss of generality since if $\overline{R}$ is a real closure of the ordered field $R$ and $n \geq 1$, then $\widetilde{R}^n = \widetilde{\overline{R}}^n$ as we have defined it.

The motivation for this paper came from Bruce Reznick who conjectured that Theorem I (as stated for $\mathbb{R}$) failed for nonarchimedean $R$ during a talk on blenders in May, 1989. A few weeks later Tom Craven pointed out the counterexample in §1 for the case $R = \mathbb{Q}$. The author wishes to thank both Reznick and Craven for their inspiration and an anonymous referee for suggesting some clarifications and simplifications which have been incorporated.

## 1. A COUNTEREXAMPLE

Suppose $R \neq \mathbb{R}$ and that $\alpha$ is a (finite) Dedekind cut of $R$ which is not represented by an element of $R$. In keeping with notation from [7, §2], we will represent $\alpha$ by its *left set* $\alpha_l$ and its *right set* $\alpha_r$. Thus $\alpha_l$ and $\alpha_r$ are

We use the word "interval" to include singletons, half-infinite, half-open, closed, infinite, and half-infinite intervals—in short, any semialgebraically connected subset of $R$.

two nonempty open half-infinite intervals[1] with the properties that $\alpha_l \le \alpha_r$, meaning that $x \in \alpha_l, y \in \alpha_r \Rightarrow x \le y$, and that $\alpha_l \cup \alpha_r = R$. In the case at hand, $\alpha_l < \alpha_r$.

Now consider the upper half-plane $H \subset R^2$ whose boundary is the nonexistent line $x_2 = \alpha x_1$. Thus $H$ consists of all points $(0, x_2)$ with $x_2 \ge 0$ plus all $(x_1, tx_1) \in R^2$ such that

(1) $$x_1 \ge 0 \text{ and } t \in \alpha_r \quad \text{or} \quad x_1 \le 0 \text{ and } t \in \alpha_l.$$

$H$ is closed and convex, but there is no way to separate $H$ from *any* $\mathbf{p} \notin H$ using a line. To see this, suppose $t \in \alpha_r$ and consider the line $x_2 = tx_1 + b$. Using the fact that $\alpha_r$ has no left endpoint, we may choose $x_1$ positive and large enough so that $tx_1 + b = t'x_1$ with $t'$ still in $\alpha_r$. Thus $(x_1, tx_1 + b) = (x_1, t'x_1) \in H$. The same idea works for $t \in \alpha_l$ and the lines $x_1 = c$. In other words, *every* line intersects $H$. This shows that Theorem I as stated for $\mathbb{R}$ needs some modification for other $R$.

## 2. A proof of Theorem I for $\mathbb{R}$

This section contains a very simple proof of Theorem I for $\mathbb{R}$ based upon the following lemma:

**Lemma 1.** *Let* $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ *and suppose that* $\mathbf{p} \cdot \mathbf{q} < \|\mathbf{p}\|^2$. *Then for all sufficiently small* $t > 0$ *we have*

(2) $$\|t\mathbf{q} + (1-t)\mathbf{p}\|^2 < \|\mathbf{p}\|^2.$$

*Proof*. Use the fact that

$$\|t\mathbf{q} + (1-t)\mathbf{p}\|^2 - \|\mathbf{p}\|^2 = t\{t(\|\mathbf{q}\|^2 + \|\mathbf{p}\|^2 - 2\mathbf{q} \cdot \mathbf{p}) + 2(\mathbf{q} \cdot \mathbf{p} - \|\mathbf{p}\|^2)\}. \quad \square$$

*Proof of Theorem* I *over* $\mathbb{R}$. We may assume that $\mathbf{p} = \mathbf{0}$. Let $\mathbf{a} \in C$ be a point with minimal distance to $\mathbf{0}$, i.e., with minimal Euclidean norm. The existence of $\mathbf{a}$ is implied by the completeness of $\mathbb{R}^n$. Lemma 1 and convexity imply that

(3) $$\mathbf{a} \cdot \mathbf{x} \ge \|\mathbf{a}\|^2$$

for all $\mathbf{x} \in C$. Thus $L(\mathbf{x}) = \mathbf{a} \cdot (\mathbf{x} - \mathbf{a}/2)$ is the sought after function. $\square$

In §5 we carry out this proof for general $R$, but first we need to compactify the domain $R^n$, complete the range $R$, and define new linear functions. This is done via the real spectrum, starting in the next section.

## 3. $\widetilde{R}$-valued linear functions

The semialgebraic subsets of $R^n$ form a boolean algebra and $\widetilde{R}^n$ may be defined as the set of ultrafilters of this algebra. (Remember, we are assuming that $R$ is real closed.) We will use this definition exclusively and denote points of $\widetilde{R}^n$ by lower case Greek letters. The topology on $\widetilde{R}^n$ is that generated by the sets

$$\widetilde{U} := \{\alpha \in \widetilde{R}^n \mid U \in \alpha\}$$

for open semialgebraic sets $U \subset R^n$. $\widetilde{R}^n$ is quasicompact but not Hausdorff. If $\beta$ is in the closure of $\alpha$ we say that $\alpha$ *specializes* to $\beta$ and write $\alpha \to \beta$. The concepts of closed and bounded for points are defined by

**Definition 1.** A point $\alpha \in \widetilde{R}^n$ is *closed* if it has no proper specializations and *bounded* if it contains a bounded set.

If $A \subseteq R^n$ and $B \subseteq R$ are semialgebraic sets, and $f$ is a map with semialgebraic graph, then $f(A)$ and $f^{-1}(B)$ are also semialgebraic [2]. It is easy to verify that

$$\{f(A) \mid A \in \alpha\} \tag{4}$$

generates an ultrafilter if $\alpha \in \widetilde{R}^n$ is an ultrafilter. Thus $f$ induces a function $\widetilde{f} : \widetilde{R}^n \to \widetilde{R}$ where $\widetilde{f}(\alpha)$ is defined to be the ultrafilter in $\widetilde{R}$ generated by (4). If $f$ is also continuous, then $\widetilde{f}$ is continuous as a function from $\widetilde{R}^n$ to $\widetilde{R}$. Details appear in [1], [2, 7.2.8], and [2, 7.3].

As a particular example of this, fix $\mathbf{x} \in R^n$ and consider the dual linear function

$$L_{\mathbf{x}}(\mathbf{y}) := \mathbf{x} \cdot \mathbf{y}. \tag{5}$$

Then $L_{\mathbf{x}}$ is a continuous semialgebraic function, so we may extend it to a function

$$\widetilde{L}_{\mathbf{x}} : \widetilde{R}^n \to \widetilde{R} \tag{6}$$

whose value at the ultrafilter $\alpha$ is computed by considering each semialgebraic subset $A \in \alpha$ and dotting each point in $A$ with $\mathbf{x}$. This produces a semialgebraic subset of $R$, and the set of these sets forms an ultrafilter $\widetilde{L}_{\mathbf{x}}(\alpha)$ in $\widetilde{R}$.

This process may be dualized to produce a function

$$L_\alpha : R^n \to \widetilde{R}, \quad \mathbf{x} \mapsto L_{\mathbf{x}}(\alpha). \tag{7}$$

**Definition 2.** An *$\widetilde{R}$-valued linear function* is a map $L : R^n \to \widetilde{R}$ of the form $L(\mathbf{x}) = L_\alpha(\mathbf{x} - \mathbf{b})$ with $L_\alpha$ as in (7) and $\mathbf{b} \in R^n$.

We remark that extending $L_a$ to all of $\widetilde{R}^n$, i.e., defining an inner product on $\widetilde{R}^n$, is too much to ask for. Indeed, for $n = 1$ this would define a multiplication on $\widetilde{R}$, and this is hopeless if $R$ is nonarchimedean. To see why, let $\eta$ be the ultrafilter of semialgebraic sets which span the gap between the positive infinitesimals and the positive noninfinitesimal elements (with respect to $\mathbb{Q}$), and let $\zeta$ be the ultrafilter spanning the gap between the positive finite and the positive infinite elements. The product of any set from $\eta$ with any set from $\zeta$ contains all positive, finite, noninfinitesimal elements. Thus " $\eta \cdot \zeta$ " is contained in an infinite number of ultrafilters.

Also, the function $L_\eta(x)$ jumps from $\eta$ to $\zeta$ as the argument $x \in R$ crosses the gap $\zeta$. $L_\eta$ is therefore *not a slice* [7] and certainly cannot be extended to a continuous function $\widetilde{R}^n \to \widetilde{R}$. In general, $L_\alpha$ need not even be continuous as function from $R^n \to \widetilde{R}$ as can be seen by taking $n = 1$ and $\alpha = +\infty$. But we do have

**Proposition 1.** *If $\alpha$ is closed and bounded, then the function $L$ in Definition 2 is continuous. For any $\alpha$ we have the following "sublinearity" property: If $s_1 \le L(\mathbf{x}) \le s_2$ and $t_1 \le L(\mathbf{y}) \le t_2$ for $s_1, s_2, t_1, t_2 \in R$, then $s_1 + s_2 \le L(\mathbf{x}+\mathbf{y}) \le t_1 + t_2$.*

*Proof*. We may assume that $L = L_\alpha$. A subbasic open set in $\widetilde{R}$ consists of the set $\widetilde{I}$ of ultrafilters containing an open interval $I = (a, b) \subset R$. We have

(8) $$L_\alpha(\mathbf{x}) \in \widetilde{I} \text{ if and only if } \mathbf{x}\cdot A \subseteq I \text{ for some } A \in \alpha.$$

Now, if $L_\alpha(\mathbf{x}) \in \widetilde{I}$, there must be some closed bounded $B \in \alpha$ such that $\mathbf{x}\cdot B$ is a finite union of (necessarily closed and bounded) subintervals of $(a, b)$. For otherwise every closed bounded set $B \in \alpha$ would contain points in the semialgebraic set

$$K := \{\mathbf{y} \mid \mathbf{x}\cdot\mathbf{y} \le a \text{ or } \mathbf{x}\cdot\mathbf{y} \ge b\}.$$

Since every closed semialgebraic set in $\alpha$ contains a closed bounded semialgebraic set in $\alpha$, the set of $B \cap K$ with $B \in \alpha$ and $B$ closed would then have the finite intersection property, from which the existence of a specialization $\beta$ of $\alpha$ with $L_\beta(\mathbf{x}) \notin (a, b)$ would follow. Since $L_\beta(\mathbf{x}) \ne L_\alpha(\mathbf{x})$, we see $\beta \ne \alpha$, contradicting the assumption that $\alpha$ is closed.

We have seen that there is a closed bounded $B \in \alpha$ such that $\mathbf{x}\cdot B \subseteq [c, d]$ with

(9) $$a < c < d < b.$$

Using (9) and a bound on the norm of points in $B$ it is straightforward to find an $\varepsilon > 0$ so that $\mathbf{z}\cdot B \subseteq I$ whenever $\|\mathbf{x}-\mathbf{z}\| < \varepsilon$. This implies $L_\alpha(\mathbf{z}) \in \widetilde{I}$ whenever $\|\mathbf{x}-\mathbf{z}\| < \varepsilon$ and establishes continuity.

For the last statement, we note that there is a set $A \in \alpha$ such that for every $\mathbf{a} \in A$ we have $s_1 \le \mathbf{a}\cdot\mathbf{x} \le s_2$ and $t_1 \le \mathbf{a}\cdot\mathbf{y} \le t_2$ and hence $s_1 + s_2 \le \mathbf{a}\cdot(\mathbf{x}+\mathbf{y}) \le t_1 + t_2$. □

Finally, we point out that if $R$ is an arbitrary ordered field and $\alpha = \mathbf{x}$ is a rational point in $\overline{R}^n$ over a real closure of $R$, then $L_\alpha$ is just the restriction of the dot product with $\mathbf{x}$ to $R^n$.

## 4. Some groundwork

The purpose of this section is to recall some key results on $\widetilde{R}$ from [7] and to isolate a slightly technical but trivial lemma on abstract functions.

In [7, §2] it is shown that the points $\alpha \in \widetilde{R}$ for $R$ real closed may be represented as pairs $(\alpha_l, \alpha_r)$ of subsets of $R$, called *slices*, satisfying $\alpha_l \le \alpha_r$

and $\alpha_l \cup \alpha_r = R$ . Specifically, for a point $x \in R$ we have

$$x \in \alpha_l \text{ if and only if } (-\infty, x] \in \alpha, \qquad (10)$$
$$x \in \alpha_r \text{ if and only if } [x, +\infty) \in \alpha.$$

The crucial result for us is

**Proposition 2.** *$\widetilde{R}$ is totally ordered with*

$$\alpha \leq \beta \textit{ if and only if } \alpha_l \subseteq \beta_l \textit{ and } \beta_r \subseteq \alpha_r. \qquad (11)$$

*Every subset of $\widetilde{R}$ has both a supremum and an infimum in $\widetilde{R}$.*

*Proof.* [7, §2]. □

The technical result we need, which we state in more generality than necessary, is a souped-up version of the result that the extension of a continuous semialgebraic function to $\widetilde{R}^n$ assumes a minimum on a closed subset: Let $K \subseteq \widetilde{R}^n$ be closed and let $f : R^n \to R$ be a continuous semialgebraic function. Let

$$\lambda := \inf_{\alpha \in K} \widetilde{f}(\alpha).$$

In addition, let $\{g_s\}_{s \in \mathscr{S}}$ be a family of continuous semialgebraic functions from $R^n$ to $R$ and let $\{\kappa_s\}_{s \in \mathscr{S}}$ be a family of closed points in $\widetilde{R}$ such that for any finite subset $s_1, \dots, s_m \in \mathscr{S}$ and any $\varepsilon \in R$ with $\varepsilon \geq \lambda$ there are points $\alpha \in K$ such that simultaneously $\widetilde{f}(\alpha) \leq \varepsilon$ and $\widetilde{g}_{s_i}(\alpha) \geq \kappa_{s_i}$ for $i = 1, \dots, m$.

**Lemma 2.** *Under the hypotheses just given there is a closed point $\zeta \in K$ with $\widetilde{f}(\zeta) = \lambda$ and $g_s(\zeta) \geq \kappa_s$ for all $s \in \mathscr{S}$.*

*Proof.* The sets

$$K_{\varepsilon, s_1, \dots, s_m} := \{ \alpha \in K \mid \widetilde{f}(\alpha) \leq \varepsilon \text{ and } \widetilde{g}_{s_i}(\alpha) \geq \kappa_{s_i} \text{ for } i = 1, \dots, m \}$$

are closed and have the finite intersection property. Therefore there is a closed point $\zeta$ in their intersection. □

## 5. Proof of Theorem I

Let $C \subset R^n$ be closed and convex and assume that $0 \notin C$ with the intention of proving Theorem I. Let

$$N^2(\mathbf{x}) = \|\mathbf{x}\|^2 = \sum_{i=1}^{n} x_i^2$$

be the square of the Euclidean norm, which is a continuous semialgebraic function. The following is a version of Lemma 1 from §2:

**Lemma 3.** *Let $\mathbf{p}$ and $\mathbf{q}_1, \dots, \mathbf{q}_m$ be finitely many points in $C$. Then there is a point $\mathbf{a} \in C$ such that $N(\mathbf{a}) \leq N(\mathbf{p})$ and such that for $i = 1, \dots, m$ we have*

$$\mathbf{q}_i \cdot \mathbf{a} \geq N^2(\mathbf{a}). \qquad (12)$$

*Proof.* We might as well assume that $C$ is the convex hull of the $\mathbf{q}_i$. This is a closed bounded semialgebraic set, so there is a point $\mathbf{a}$ in it of minimal norm [2, 2.5.8]. The calculation used to prove Lemma 1 is valid over any $R$, from which (12) follows. □

The same argument works in the case of an arbitrary closed bounded semialgebraic set $C$ and (using the fact that a semialgebraic function achieves a maximum on $C$ as well) yields

**Lemma 4.** *If $C$ is a closed bounded semialgebraic set in $R^n$ which does not contain $0$, then there is a vector $\mathbf{x} \in R^n$ such that $\mathbf{x} \cdot C$ is contained in a closed interval $[a, b]$ with $0 < a \leq b < \infty$.* □

*Proof of Theorem* I. Let $K$ be the closure of $C$ in $\widetilde{R}^n$ and let

$$\lambda := \inf_{\mathbf{q} \in C} N(\mathbf{q}).$$

Since $C$ is closed, we have $0^+ < \lambda$, so we may choose a $z \in R$ with $0 < z \leq \lambda$. We now consider the semialgebraic function $N$, the family $\{L_\mathbf{q}\}_{\mathbf{q} \in C}$ of semialgebraic functions, and the constant family $\kappa_\mathbf{q} = z^2$. If $\varepsilon \geq \lambda$, there is a point $\mathbf{p} \in C$ with $N(\mathbf{p}) \leq \varepsilon$. Lemma 3 says that this setup satisfies the hypotheses of Lemma 2, so we find a closed $\zeta \in K$ such that

$$\widetilde{N}(\zeta) = \lambda \quad \text{and} \quad \widetilde{L}_\mathbf{q}(\zeta) \geq z^2 \text{ for all } \mathbf{q} \in C. \tag{13}$$

Let $\widetilde{L}_\zeta$ be defined as in (7). From (13) we see both that $\zeta$ is bounded and that $\widetilde{L}_\zeta(\mathbf{q}) \geq z^2$ for $\mathbf{q} \in C$. By Proposition 1 the function

$$L(\mathbf{x}) = \widetilde{L}_\zeta(\mathbf{x} - \mathbf{b})$$

fulfills the requirements of Theorem I for any $\mathbf{b}$ with $-z^2/2 < \widetilde{L}_\zeta(-\mathbf{b}) < 0$.

To see that such a $\mathbf{b}$ exists, note that $\zeta$ contains a closed bounded $B$ which does not contain $0$. We now apply Lemma 4 to $B$ and let $\mathbf{b} = -\frac{z^2}{2b}\mathbf{x}$ with $\mathbf{x}$ and $b$ as in the lemma. □

## 6. Slices and rips

In the description of $\widetilde{R}$ from [7], the set $\alpha_l$ is called the *left set* of $\alpha$ and consists of all $x \in R$ with $x \leq \alpha$, while $\alpha_r$ is called the *right set* of $\alpha$ and consists of all $x$ with $\alpha_r \leq x$. If we try to add two points $\alpha$ and $\beta$ by adding their left and right sets, we see that

$$\alpha_l + \beta_l \leq \alpha_r + \beta_r$$

and also that

$$\begin{aligned} &x \in \alpha_l + \beta_l \text{ and } y < x \Rightarrow y \in \alpha_l + \beta_l, \\ &x \in \alpha_r + \beta_r \text{ and } y > x \Rightarrow y \in \alpha_r + \beta_r, \end{aligned} \tag{14}$$

but in general there is a gap between the set sums in (14). For example, the point $2$ is in neither the sum of the left sets nor in the sum of the right sets of " $1^- + 1^+$ ". In the nonarchimedean case entire intervals may be left out.

For this reason we are led to consider more general pairs

$$\zeta = (\zeta_l, \zeta_r) \tag{15}$$

of left and right sets and define

**Definition 3.** A *rip* is a pair (15) satisfying

$$\begin{gathered} \zeta_l \le \zeta_r, \\ x \in \zeta_l \text{ and } y < x \Rightarrow y \in \zeta_l, \text{ and} \\ x \in \zeta_r \text{ and } y > x \Rightarrow y \in \zeta_r. \end{gathered} \tag{17}$$

Thus points in $\widetilde{R}$ are special types of rips. If $\zeta$ and $\xi$ are rips, we define

$$\zeta + \xi = (\zeta_l + \xi_l, \zeta_r + \xi_r),$$

which is again a rip. Next, we borrow from John Conway's philosophy (used to define the ordering on the surreal numbers in [JC, Chapter 1]) and define $\zeta \le \xi$ *unless there is an obstruction to this inequality.* An obstruction is a point $x \in R$ with either $\xi < x \le \zeta$ or $\xi \le x < \zeta$. We interpret $\xi \le x$ to mean that $x \in \xi_r$ and $\xi < x$ to mean that $x \in \xi_r \setminus \xi_l$. There is no such obstruction $x$ if

$$\zeta_l \cap \xi_r \setminus \xi_l = \zeta_l \cap \xi_r \setminus \zeta_r = \varnothing,$$

which may be rephrased as

**Definition 4.** Let $\zeta$ and $\xi$ be two rips. We define $\zeta \le \xi$ if and only if and $(\zeta_l \cap \xi_r) \subseteq (\zeta_r \cap \xi_l)$.

**Proposition 3.** *The "$\le$"-relation on rips extends the total ordering on $\widetilde{R}$. Given two rips $\zeta$ and $\xi$, either $\zeta \le \xi$, or $\xi \le \zeta$ or both.*

*Proof*. The first statement follows from the development of Definition 4 and can be checked by looking at cases. To prove the second statement, we need to exclude the possibility that there are $x, y \in R$ with

$$\begin{gathered} x \in \zeta_l \cap \xi_r \text{ and } x \notin \zeta_r \cap \xi_l, \\ y \in \zeta_r \cap \xi_l \text{ and } y \notin \zeta_l \cap \xi_r. \end{gathered} \tag{18}$$

But if (18) holds, then $x \in \zeta_l$ and $y \in \zeta_r$ imply that $x \le y$, while $x \in \xi_r$ and $y \in \xi_l$ imply that $x \ge y$. Hence $x = y$, but now (18) is clearly contradictory. □

## 7. The metric $\mu$

To define $\mu$, consider $\alpha, \beta \in \widetilde{R}^n$. If $A \in \alpha$ and $B \in \beta$, we define

$$d(A, B) := \inf_{\substack{\mathbf{a} \in A \\ \mathbf{b} \in B}} \|\mathbf{a} - \mathbf{b}\|. \tag{19}$$

*Note that the infimum in* (19) *is to be taken not in $R$ (where it need not exist) but in $\widetilde{R}$.* We then go on to define

$$\mu(\alpha, \beta) := \sup_{\substack{A\in\alpha \\ B\in\beta}} d(A, B) \tag{20}$$

where the supremum is taken in $\widetilde{R}$ as well. Interpreting the values of $\mu$ as rips for the triangle inequality, we have:

**Theorem II.** *The function* $\mu\colon \widetilde{R}^n \times \widetilde{R}^n \to \widetilde{R}$ *is a positive definite symmetric function satisfying the triangle inequality. Moreover, if* $\mathbf{a}, \mathbf{b} \in R^n \subset \widetilde{R}^n$, *then* $\mu(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|$.

*Proof*. Symmetry is obvious from the definition as is the fact that $\mu(\alpha, \beta) \geq 0$. For the rest of the proof we let $\alpha, \beta, \gamma \in \widetilde{R}^n$.

Suppose $\mu(\alpha, \beta) = 0$. Since $\{x \in R \mid x > 0\}$ has the infimum $0^+$ in $\widetilde{R}$, every set $A \in \alpha$ must have zero distance to every set $B \in \beta$. In other words, $A \cap B \neq \emptyset$ for every $A \in \alpha$ and $B \in \beta$. Thus $\alpha \cup \beta$ is a filter, which implies $\alpha = \beta$ since both are ultrafilters. Thus $\mu$ is positive definite.

The triangle inequality states that

$$\mu(\alpha, \gamma) \leq \mu(\alpha, \beta) + \mu(\beta, \gamma).$$

To verify this we need to show that

$$\big(\mu(\alpha, \gamma)_l \cap (\mu(\alpha, \beta)_r + \mu(\beta, \gamma)_r)\big) \subseteq \big(\mu(\alpha, \gamma)_r \cap (\mu(\alpha, \beta)_l + \mu(\beta, \gamma)_l)\big). \tag{21}$$

So suppose there is a point $z$ in the left-hand set given in (21). Since $z \in \mu(\alpha, \beta)_r + \mu(\beta, \gamma)_r$, there are points $x, y \in R$ such that $x + y = z$ and such that given any $A \in \alpha$, $B \in \beta$, and $C \in \gamma$ there are points $\mathbf{a} \in A$, $\mathbf{b} \in B$ and $\mathbf{b}' \in B$, $\mathbf{c} \in C$ with

$$\|\mathbf{a} - \mathbf{b}\| \leq x \quad \text{and} \quad \|\mathbf{b}' - \mathbf{c}\| \leq y. \tag{22}$$

Fixing $A$, $B$, and $C$, consider the set

$$B' := \{\, \mathbf{b} \in B \mid d(A, \{\mathbf{b}\}) \leq x\}.$$

We must have $B' \in \beta$, for otherwise its complement would be in $\beta$ and (22) could not hold for any $\mathbf{b} \in B'$. Now consider the set

$$B'' := \{\, \mathbf{b}' \in B' \mid d(\{\mathbf{b}'\}, C) \leq y\,\}. \tag{23}$$

Again, we must have $B'' \in \beta$. But this implies that $d(A, C) \leq z$ for any $A \in \alpha$ and $C \in \gamma$. Thus we conclude that $z \in \mu(\alpha, \gamma)_r$.

By assumption, $z \in \mu(\alpha, \gamma)_l$. Thus there are sets $A \in \alpha$ and $C \in \gamma$ with $d(A, C) \geq z$. Fix these sets. Suppose there were an $E \in \beta$ with $d(A, \{\mathbf{e}\}) < x$ for all $\mathbf{e} \in B$. By intersecting $E$ with $B''$ from (23) we would obtain $d(A, C) < z$. Thus the set

$$E := \{\, \mathbf{b} \in B' \mid d(A, \{\mathbf{b}\}) = x\,\}$$

is in $\beta$, from which we see that $x \in \mu(\alpha, \beta)_l$. Similarly, $y \in \mu(\beta, \gamma)_l$, and so $z = x + y \in \mu(\alpha, \beta)_l + \mu(\beta, \gamma)_l$, establishing (21). □

**Some examples.** In order to illustrate $\mu$, we state (without proof) some values in a few simple cases. First, a table of values for pairs of points in $\widetilde{R}$ with the first point specializing to 1 and the second to 2:

$$\begin{array}{ccc} \mu(1^-, 2^-) = 1^-, & \mu(1, 2^-) = 1^-, & \mu(1^+, 2^-) = 1^+, \\ \mu(1^-, 2) = 1^+, & \mu(1, 2) = 1, & \mu(1^+, 2) = 1^-, \\ \mu(1^-, 2^+) = 1^+, & \mu(1, 2^+) = 1^+, & \mu(1^+, 2^+) = 1^-. \end{array}$$

For a slightly more complicated case, consider the two-dimensional point $\alpha \in \widetilde{\mathbb{R}}^2$ consisting of all semialgebraic subsets $A$ containing a set of the form $\{(x, e^x) \mid 0 < x < \varepsilon\}$ for some $\varepsilon > 0$. Let $\beta$ be the one-dimensional point consisting of all semialgebraic sets containing some piece of the algebraic half-branch $y = 0$, $x > 0$ at $(0, 0)$. Then $\mu(\alpha, \beta) = 1^+$.

In order to clarify the nature of $\mu$, we point out

**Proposition 4.** *Suppose $\alpha, \beta \in \widetilde{R}^n$ and $\alpha$ is bounded. Then $\mu(\alpha, \beta) = 0^+$ if and only if $\alpha$ and $\beta$ are distinct but have a common specialization.*

*Proof.* Suppose $\mu(\alpha, \beta) = 0^+$. Let $A \in \alpha$ be bounded, let $B \in \beta$ be arbitrary, and let $\overline{A}$ and $\overline{B}$ be their closures in $R^n$. If we fix $\mathbf{x}$, then the function of $\mathbf{x}$ which measures the distance to $\overline{B}$ is a continuous semialgebraic function [2 2.5.8]. This takes a minimum on the closed bounded set $\overline{A}$. If $\mu(\alpha, \beta) = 0^+$, this minimum cannot be strictly positive, so $\overline{A} \cap \overline{B} \neq \varnothing$. We conclude that

$$\{\overline{A} \cap \overline{B} \mid A \in \alpha, B \in \beta\} \tag{24}$$

has the finite intersection property and is contained in at least one ultrafilter $\gamma$. Since any closed set in $\alpha$ is in $\gamma$, $\alpha \to \gamma$. Similarly, $\beta \to \gamma$.

If $\alpha$ and $\beta$ have no common specialization, we can find disjoint closed sets $A \in \alpha$ and $B \in \beta$ [7], and since we may choose $A$ to be bounded, it is immediate that $\mu(\alpha, \beta) \geq d(A, B) = x > 0$ for some $x \in R$, so $\mu(\alpha, \beta) > 0^+$. □

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